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Positive and negative integrable hierarchies, associated conservation laws and Darboux transformation

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ABSTRACT

Two hierarchies of integrable positive and negative lattice equations in connection with a new discrete isospectral problem are derived. It is shown that they correspond to positive and negative power expansions respectively of Lax operators with respect to the spectral parameter, and each equation in the resulting hierarchies is Liouville integrable. Moreover, infinitely many conservation laws of corresponding positive lattice equations are obtained in a direct way. Finally, a Darboux transformation is established with the help of gauge transformations of Lax pairs for the typical lattice soliton equations, by means of which the exact solutions are given.

1. Introduction

Lattice soliton theory, as a new subject, is well known to be an effective tool used for describing and explaining the nonlinear phenomena such as nonlinear optics, superconductivity, plasma physics, magnetic fluid, etc. The study of soliton equations, therefore, has always been one of the most prominent events in the field of nonlinear science during the past few years, and many lattice soliton equations have been studied systematically.

Searching for new lattice soliton equations, in lattice soliton theory, is still an important and complicated task. The discrete zero curvature representation is an effective method to generate the lattice soliton equations, and the Hamiltonian structures of the lattice soliton equations can be established by the discrete trace identity. Furthermore, the conservation laws play an important role, the existence of which for the lattice soliton equations may further confirm their integrability. As is well known, nonlinear integrable systems of lattice versions, treated as models of some physical phenomena, have become the focus of common concern in the recent decade. Lots of nonlinear integrable lattice soliton equations have been obtained and discussed systematically, for instance, the Ablowitz–Ladik lattice [1], the Toda lattice [2], the Blaszak–Marciniak lattice [3,4], and so forth [5–8].

Moreover, a considerable number of nonlinear differential–difference evolution equations are well known to be solved by means of the inverse scattering transform method [9], the bilinear transformation method of Hirota [10], the Bäcklund and Darboux transformation technique [11], and so on. The Darboux transformation (DT), which has always been used to offer the explicit solutions of the soliton equations, was first introduced in [12] in the study of Sturm–Liouville equation. Extensive applications of DT can be found in [13–21], and the references therein.

This paper is devoted to introducing a new discrete isospectral problem

$$E\psi_n = U_n(u,\lambda)\psi_n, \quad U_n(u,\lambda) = \begin{pmatrix} \lambda & 1+r_n \\ \lambda s_n & 1+s_n+r_ns_n \end{pmatrix}.$$

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By means of constructing a proper continuous time evolution equation and using the discrete zero curvature equation, two hierarchies of positive and negative lattice models [7] are presented. It is shown that the positive and negative hierarchies correspond to positive and negative power expansions with respect to spectral parameter, and, they are of rational and polynomial type equations about potentials, respectively. As typical examples, the first two nonlinear discrete evolution equations from positive and negative hierarchies are given as follows:

$$\begin{cases} r_{n_t} = 1 + r_{n+1} - (1+r_n)^2 s_{n-1}, \\ s_{n_t} = -s_{n-1} + (1+r_{n+1}) s_n^2. \end{cases} \tag{a}$$

and

$$\begin{cases} r_{n_t} = -\dfrac{1+r_{n-1}}{1+s_{n-1}+r_{n-1}s_{n-1}}, \\ s_{n_t} = \dfrac{s_{n+1}}{1+s_{n+1}+r_{n+1}s_{n+1}}. \end{cases} \tag{b}$$

It is especially worth pointing out that (a), equation of rational type version, has scarcely come forth previously. Moreover, the conservation laws of the obtained equation hierarchies are discussed. Finally, by virtue of the gauge transformation, we find a DT for spectral problem. As an application, exact solutions of (a) are given.

This paper is divided into five sections. Section 2 will be devoted to a pair of integrable positive and negative hierarchies of lattice soliton equations together with their Hamiltonian structures. In Section 3, the Conservation laws of the integrable positive lattice equations are discussed. In Section 4, a Darboux transformation is established with the help of the gauge transformations of Lax pairs for the typical (a) coming from resulting hierarchies, by means of which exact solutions are given. Finally, in Section 5, there will be some conclusions and remarks.

2. Integrable lattice models and Hamiltonian structures

First, we specify some fundamental conceptions. The shift operator E, two difference operators D and Δ are defined as follows

$$\begin{aligned} &(Ef)(n) = f(n+1), \qquad (E^{-1}f)(n) = f(n-1), \quad n \in Z. \\ &(Df)(n) = f(n+1) - f(n), \qquad (\Delta f)(n) = f(n+1) - f(n-1), \quad n \in Z \end{aligned} \tag{1}$$

where f is a lattice function from Z to R. We assume that $U_n = (r_n, s_n)^{\mathrm{T}}$, $r_n = r(n,t)$, $s_n = s(n,t)$ are real functions defined over $Z \times R$, and U_n is required to vanish rapidly at the infinity. λ is the spectral parameter and $\lambda_t = 0$.

A lattice equation

$$U_{n_t} = K(u_n, Eu_n, E^{-1}u_n, \ldots) \tag{2}$$

is said to be Lax integrable, if it can be rewritten as a compatibility condition

$$U_{n_t} = (EV_n(u,\lambda))U_n(u,\lambda) - U_n(u,\lambda)V_n(u,\lambda) \tag{3}$$

of a discrete spatial problem

$$E\psi_n = U_n(u,\lambda)\psi_n \tag{4}$$

and a corresponding continuous time evolution equations

$$\psi_{n_t} = V_n(u,\lambda)\psi_n \tag{5}$$

of a discrete spatial problem

$$E\psi_n = U_n\psi_n, \quad U_n(u,\lambda) = \begin{pmatrix} \lambda & 1+r_n \\ \lambda s_n & 1+s_n+r_ns_n \end{pmatrix}, \quad \psi_n = \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix}, \quad u_n = \begin{pmatrix} r_n \\ s_n \end{pmatrix}, \quad \lambda_t = 0. \tag{6}$$

2.1. The positive hierarchy and its Hamiltonian structure

Let

$$\Gamma_n = \begin{pmatrix} a_n & b_n \\ \lambda c_n & -a_n \end{pmatrix}. \tag{7}$$

Solving the stationary discrete zero curvature equation

$$(E\Gamma_n)U_n - U_n\Gamma_n = 0, \tag{8}$$

gives rise to

$$\begin{cases}\lambda(a_{n+1}-a_n+\lambda s_n b_{n+1}-\lambda(1+r_n)c_n)=0,\\(1+r_n)(a_{n+1}+a_n)+(1+s_n+r_ns_n)b_{n+1}-\lambda b_n=0,\\\lambda^2c_{n+1}-\lambda s_n(a_{n+1}+a_n)-\lambda(1+s_n+r_ns_n)c_n=0,\\\lambda(1+r_n)c_{n+1}-\lambda s_nb_n-(1+s_n+r_ns_n)(a_{n+1}-a_n)=0.\end{cases}\tag{9}$$

Let $a_n=\sum_{m=0}^{\infty}a_n^{(m)}\lambda^{-m},\ b_n=\sum_{m=0}^{\infty}b_n^{(m)}\lambda^{-m},\ c_n=\sum_{m=0}^{\infty}c_n^{(m)}\lambda^{-m}$. From (9) we have the following initial value

$$b_n^{(0)}=c_{n+1}^{(0)}=0,\qquad a_{n+1}^{(0)}-a_n^{(0)}=(1+r_n)c_n^{(0)}-s_nb_{n+1}^{(0)}$$

and recurrence relation

$$\begin{cases}a_{n+1}^{(m)}-a_n^{(m)}+s_nb_{n+1}^{(m)}-(1+r_n)c_n^{(m)}=0,\\(1+r_n)(a_{n+1}^{(m)}+a_n^{(m)})+(1+s_n+r_ns_n)b_{n+1}^{(m)}=b_n^{(m+1)},\\-s_n(a_{n+1}^{(m)}+a_n^{(m)})-(1+s_n+r_ns_n)c_n^{(m)}=-c_{n+1}^{(m+1)},\\-(1+s_n+r_ns_n)(a_{n+1}^{(m)}-a_n^{(m)})=s_nb_n^{(m+1)}-(1+r_n)c_{n+1}^{(m+1)}.\end{cases}\tag{10}$$

Take $a_n^{(0)}=\frac{1}{2},\ b_n^{(0)}=c_n^{(0)}=0$, require $a_j|_{[u]=0}=0$, $b_j|_{[u]=0}=0$, $c_j|_{[u]=0}=0$ $(j\geq 1)$. The first coefficients are given as follows:

$$\begin{aligned}&a_n^{(1)}=-(1+r_n)s_{n-1},\qquad b_n^{(1)}=(1+r_n),\qquad c_{n+1}^{(1)}=s_n,\\&a_n^{(2)}=s_{n-1}^2+2r_ns_{n-1}^2+r_n^2s_{n-1}^2-r_ns_{n-2}-r_{n+1}s_{n-1}-s_{n-2}-s_{n-1},\\&b_n^{(2)}=1+r_{n+1}-(1+r_n)^2s_{n-1},\\&c_{n+1}^{(2)}=s_{n-1}-(1+r_{n+1})s_n^2,\ \ldots.\end{aligned}$$

By this way, the recursion relation (10) determines uniquely $a_j, b_j, c_j,\ j\geq 1$.
Denote

$$\Gamma_n^{\{m\}}=\sum_{i=0}^{m}\begin{pmatrix}a_n^{(i)}\lambda^{m-i} & b_n^{(i)}\lambda^{m-i}\\ c_n^{(i)}\lambda^{m-i+1} & -a_n^{(i)}\lambda^{m-i}\end{pmatrix},\quad m\geq 0.\tag{11}$$

Direct calculation reads

$$(E\Gamma_n^{\{m\}})U_n-U_n\Gamma_n^{\{m\}}=\begin{pmatrix}0 & b_n^{(m+1)}\\ -\lambda c_{n+1}^{(m+1)} & s_nb_n^{(m+1)}-(1+r_n)c_{n+1}^{(m+1)}\end{pmatrix}.\tag{12}$$

Then the discrete zero curvature equation admits the following positive hierarchy

$$u_{n_{t_m}}=\begin{pmatrix}r_n\\ s_n\end{pmatrix}_{t_m}=\begin{pmatrix}b_n^{(m+1)}\\ -c_{n+1}^{(m+1)}\end{pmatrix}.\tag{13}$$

When $m=0$, system (13) reduces to

$$\begin{cases}r_{n_{t_0}}=1+r_n,\\ s_{n_{t_0}}=-s_n.\end{cases}\tag{14}$$

When $m=1$, system (13) reduces to

$$\begin{cases}r_{n_{t_1}}=1+r_{n+1}-(1+r_n)^2s_{n-1},\\ s_{n_{t_1}}=-s_{n-1}+(1+r_{n+1})s_n^2.\end{cases}\tag{15}$$

Accordingly, when $m=1$, the t-part of Lax pairs for this equation is as follows

$$\Gamma_n^{\{1\}}=\begin{pmatrix}\frac{1}{2}\lambda-(1+r_n)s_{n-1} & 1+r_n\\ s_{n-1}\lambda & -\frac{1}{2}\lambda+(1+r_n)s_{n-1}\end{pmatrix}.\tag{16}$$

To establish the Hamiltonian structure for system (13), we define

$$V_n=\Gamma_nU_n^{-1}=\begin{pmatrix}\frac{a_n+(s_n+r_ns_n)a_n}{\lambda}-s_nb_n & b_n-\frac{(1+r_n)a_n}{\lambda}\\ c_n+(s_n+r_ns_n)c_n+s_na_n & -a_n-(1+r_n)c_n\end{pmatrix}\tag{17}$$

and $\langle A, B\rangle = \mathrm{Tr}(AB)$, where A and B are the same order square matrices. We have

$$\frac{\partial U_n}{\partial \lambda} = \begin{pmatrix} 1 & 0 \\ s_n & 0 \end{pmatrix}, \qquad \frac{\partial U_n}{\partial r_n} = \begin{pmatrix} 0 & 1 \\ 0 & s_n \end{pmatrix}, \qquad \frac{\partial U_n}{\partial s_n} = \begin{pmatrix} 0 & 0 \\ \lambda & 1+r_n \end{pmatrix}.$$

Hence

$$\left\langle V_n, \frac{\partial U_n}{\partial \lambda}\right\rangle = \frac{a_n}{\lambda}, \qquad \left\langle V_n, \frac{\partial U_n}{\partial r_n}\right\rangle = c_n, \qquad \left\langle V_n, \frac{\partial U_n}{\partial s_n}\right\rangle = b_{n+1}.$$

From the discrete trace identity

$$\frac{\delta}{\delta u_n}\sum_{k\in Z}\left\langle V_n, \frac{\partial U_n}{\partial \lambda}\right\rangle(k) = \left(\lambda^{-\varepsilon}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\varepsilon}\right)\left\langle V_n, \frac{\partial U_n}{\partial u_n^i}\right\rangle, \quad i=1,2. \tag{18}$$

We have

$$\begin{pmatrix} \dfrac{\delta}{\delta r_n} \\ \dfrac{\delta}{\delta s_n} \end{pmatrix}\sum_{k\in Z}\frac{a_n}{\lambda}(k) = \lambda^{-\varepsilon}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\varepsilon}\begin{pmatrix} c_n \\ b_{n+1} \end{pmatrix}.$$

Comparison of the coefficient of λ^{-m-1} yields

$$\begin{pmatrix} \dfrac{\delta}{\delta r_n} \\ \dfrac{\delta}{\delta s_n} \end{pmatrix}\sum_{k\in Z}a_n^{(m)}(k) = (\varepsilon - m)\begin{pmatrix} c_n^{(m)} \\ b_{n+1}^{(m)} \end{pmatrix}.$$

Taking $m=0$ gives $\varepsilon = 0$. Thus,

$$u_{t_m} = \begin{pmatrix} r_n \\ s_n \end{pmatrix}_{t_m} = J_1\frac{\delta H_n^{(m+1)}}{\delta u_n} = J_1\begin{pmatrix} c_n^{(m+1)} \\ b_{n+1}^{(m+1)} \end{pmatrix}, \quad m \geq 0, \tag{19}$$

where

$$J_1 = \begin{pmatrix} 0 & E^{-1} \\ -E & 0 \end{pmatrix}.$$

and

$$H_n^{(m+1)} = \sum_{k\in Z} -\frac{a_n^{(m+1)}}{m+1}(k), \quad m\geq 0, \qquad H_n^0 = \sum_{k\in Z}\ln(1+s_n+r_ns_n)(k). \tag{20}$$

Let

$$\frac{\delta H_n^{(m+1)}}{\delta u_n} = \Phi\frac{\delta H_n^{(m)}}{\delta u_n}, \quad \Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}.$$

From (10), we obtain the recurrence operator Φ as follows

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix},$$

where

$$\begin{aligned}
\Phi_{11} &= E^{-1}(1+s_n+r_ns_n) - E^{-1}s_n(1+E)(1-E)^{-1}(1+r_n),\\
\Phi_{12} &= E^{-1}s_n(1+E)(1-E)^{-1}s_n,\\
\Phi_{21} &= -E(1+r_n)(1+E)(1-E)^{-1}(1+r_n),\\
\Phi_{22} &= E(1+s_n+r_ns_n) + E(1+r_n)(1+E)(1-E)^{-1}s_n.
\end{aligned}$$

So the system (19) can be written as

$$u_{n_{t_m}} = \begin{pmatrix} r_n \\ s_n \end{pmatrix}_{t_m} = J_1\Phi^m\begin{pmatrix} c_n^{(1)} \\ b_{n+1}^{(1)} \end{pmatrix} = J_1\Phi^m\begin{pmatrix} 1+r_n \\ -s_n \end{pmatrix}. \tag{21}$$

Let

$$K_1 = J_1\Phi = \begin{pmatrix} -(1+r_n)(1+E)(1-E)^{-1}(1+r_n) & (1+r_n)(1+E)(1-E)^{-1}s_n + (1+s_n+r_ns_n) \\ -s_n(1+E)(1-E)^{-1}(1+r_n) - (1+s_n+r_ns_n) & -s_n(1+E)(1-E)^{-1}s_n \end{pmatrix}.$$

It is easy to verify that K_1 is a skew-symmetric operator. We consider the following Poisson bracket:

$$\{\tilde{f},\tilde{g}\}_{J_1} = \left\langle \frac{\delta\tilde{f}}{\delta u_n}, J_1\frac{\delta\tilde{g}}{\delta u_n}\right\rangle = \sum_{k\in Z}\sum_{i=1}^{2}\left(J_1\frac{\delta f}{\delta u_n}\right)_i\left(\frac{\delta g}{\delta u_n}(k)\right)_i, \tag{22}$$

where $\tilde{f} = \sum_{k\in Z} f(k)$, and the variational derivative is defined by

$$\frac{\delta\tilde{f}}{\delta u_n} = \left(\frac{\delta\tilde{f}}{\delta r_n}, \frac{\delta\tilde{g}}{\delta s_n}\right)^{\mathrm{T}}, \quad \frac{\delta\tilde{f}}{\delta r_n} = \sum_{k\in Z} E^{-1}\left(\frac{\partial f}{\partial r_n^{(k)}}\right), \qquad \frac{\delta\tilde{f}}{\delta s_n} = \sum_{k\in Z} E^{-1}\left(\frac{\partial f}{\partial s_n^{(k)}}\right).$$

So we have the following assertions.

Proposition 1. *$\{\tilde{H}_n^{(m+1)}\}_{m\geq 0}$ defined by (20) forms an infinite set of conserved functionals of the hierarchy (13), and $\{\tilde{H}_n^{(m+1)}\}_{m\geq 0}$ are in involution in pairs with respect to the Poisson bracket (22)*

Proof. It is not difficult to verify that $(J_1\phi)^* = -J_1\phi$, that is $J_1\phi = \phi^*J_1$. Hence

$$\begin{aligned}\{H_n^{(m)}, H_n^{(l)}\}_{J_1} &= \left\langle \frac{\delta H_n^{(m)}}{\delta u_n}, J_1\frac{\delta H_n^{(l)}}{\delta u_n}\right\rangle = \left\langle \Phi^{m-1}\frac{\delta H_n^{(1)}}{\delta u_n}, J_1\Phi^{l-1}\frac{\delta H_n^{(1)}}{\delta u_n}\right\rangle \\ &= \left\langle \Phi^{m-1}\frac{\delta H_n^{(1)}}{\delta u_n}, \Phi^*J_1\Phi^{l-2}\frac{\delta H_n^{(1)}}{\delta u_n}\right\rangle = \left\langle \Phi^{m}\frac{\delta H_n^{(1)}}{\delta u_n}, J_1\Phi^{l-2}\frac{\delta H_n^{(1)}}{\delta u_n}\right\rangle \\ &= \{H_n^{(m+1)}, H_n^{(l-1)}\}_{J_1} = \cdots = \{H_n^{(m+1-1)}, H_n^{(1)}\}_{J_1}.\end{aligned}$$

Similarly, we can find that

$$\{H_n^{(l)}, H_n^{(m)}\}_{J_1} = \{H_n^{(m+1-1)}, H_n^{(1)}\}_{J_1}, \tag{23}$$

this implies that

$$\{H_n^{(m)}, H_n^{(l)}\}_{J_1} = -\{H_n^{(l)}, H_n^{(m)}\}_{J_1}.$$

Therefore

$$\{H_n^{(l)}, H_n^{(m)}\}_{J_1} = 0, \quad m, l \geq 1 \tag{24}$$

and

$$(H_n^{(m)})_{t_l} = \left\langle \frac{\delta H_n^{(m)}}{\delta u_n}, u_{nt_l}\right\rangle = \left\langle \frac{\delta H_n^{(m)}}{\delta u_n}, J\frac{\delta H_n^{(l)}}{\delta u_n}\right\rangle = \{H_n^{(m)}, H_n^{(l)}\}_{J_1} = 0, \quad m, l \geq 1.$$

Moreover, it is known that, if J_1 is a Hamiltonian operator, then

$$\left[J_1\frac{\delta F_n}{\delta u}, \frac{\delta G_n}{\delta u_n}\right] = J_1\frac{\delta\{F_n, G_n\}_{J_1}}{\delta u_n},$$

where the commutator is defined by

$$[X, Y] := \frac{\partial}{\partial\varepsilon}(X(u+\varepsilon Y) - Y(u+\varepsilon X))|_{\varepsilon=0}.$$

From Eq. (24), we have

$$\left[J_1\frac{\delta H_n^{(m)}}{\delta u}, J_1\frac{\delta H_n^{(l)}}{\delta u_n}\right] = J_1\frac{\delta\{H_n^{(m)}, H_n^{(l)}\}_{J_1}}{\delta u_n} = 0, \quad m, l \geq 0. \quad \square$$

Theorem 1. *The lattice soliton equations in (13) or the discrete Hamiltonian equations in (19) are all discrete Liouville integrable systems.*

2.2. The negative hierarchy and its Hamiltonian structure

Let

$$\Upsilon_n = \begin{pmatrix} A_n & B_n \\ \lambda C_n & -A_n \end{pmatrix}, \tag{25}$$

solving the stationary discrete zero curvature equation

$$(E\Upsilon_n)U_n - U_n\Upsilon_n = 0. \tag{26}$$

Let $A_n = \sum_{m=0}^{\infty} A_n^{(m)}\lambda^m$, $B_n = \sum_{m=0}^{\infty} B_n^{(m)}\lambda^m$, $C_n = \sum_{m=0}^{\infty} C_n^{(m)}\lambda^m$ in (9) yields

$$\begin{cases} A_{n+1}^{(m)} - A_n^{(m)} + s_n B_{n+1}^{(m)} - (1+r_n)C_n^{(m)} = 0, \\ (1+r_n)(A_{n+1}^{(m+1)} + A_n^{(m+1)}) + (1+s_n+r_ns_n)B_{n+1}^{(m+1)} = B_n^{(m)}, \\ -s_n(A_{n+1}^{(m+1)} + A_n^{(m+1)}) - (1+s_n+r_ns_n)C_n^{(m+1)} = -C_{n+1}^{(m)}, \\ -(1+s_n+r_ns_n)(A_{n+1}^{(m+1)} - A_n^{(m+1)}) = s_nB_n^{(m)} - (1+r_n)C_{n+1}^{(m)}, \\ A_n^{(0)} = -\dfrac{1}{2}, \qquad B_n^{(0)} = -\dfrac{1+r_{n-1}}{1+s_{n-1}+r_{n-1}s_{n-1}}, \qquad C_n^{(0)} = \dfrac{s_n}{1+s_n+r_ns_n}, \\ A_n^{(1)} = -\dfrac{(1+r_{n-1})s_n}{(1+s_{n-1}+r_{n-1}s_{n-1})(1+s_n+r_ns_n)}, \\ B_{n+1}^{(1)} = \dfrac{1+r_{n-1}}{(1+s_{n-1}+r_{n-1}s_{n-1})(1+s_n+r_ns_n)^2} - \dfrac{(1+r_n)(1+r_{n-1})s_n}{(1+s_{n-1}+r_{n-1}s_{n-1})(1+s_n+r_ns_n)^2}, \\ C_n^{(1)} = \dfrac{s_{n+1}}{(1+s_{n+1}+r_{n+1}s_{n+1})(1+s_n+r_ns_n)^2} - \dfrac{(1+r_{n-1})s_n^2}{(1+s_{n-1}+r_{n-1}s_{n-1})(1+s_n+r_ns_n)^2}. \end{cases}$$

Denote

$$W_n^{\{m\}} = \sum_{i=0}^{m} \begin{pmatrix} A_n^{(i)}\lambda^{m-i} & B_n^{(i)}\lambda^{m-i} \\ C_n^{(i)}\lambda^{m-i+1} & -A_n^{(i)}\lambda^{m-i} \end{pmatrix}.$$

Direct calculation reads

$$(EW_n^{\{m\}})U_n - U_nW_n^{\{m\}} = \begin{pmatrix} 0 & -B_n^{(m)} \\ \lambda C_{n+1}^{(m)} & s_nC_{n+1}^{(m)} - (1+r_n)B_n^{(m)} \end{pmatrix}. \tag{27}$$

Then the discrete zero curvature equation admits the following negative hierarchy

$$U_{n_{t_m}} = \begin{pmatrix} r_n \\ s_n \end{pmatrix}_{t_m} = \begin{pmatrix} -B_n^{(m)} \\ C_{n+1}^{(m)} \end{pmatrix}, \tag{28}$$

when $m = 0$, system (28) reduces to

$$\begin{cases} r_{n_{t_0}} = -\dfrac{1+r_{n-1}}{1+s_{n-1}+r_{n-1}s_{n-1}}, \\ s_{n_{t_0}} = \dfrac{s_{n+1}}{1+s_{n+1}+r_{n+1}s_{n+1}}. \end{cases} \tag{29}$$

Accordingly, when $m = 0$, the t-part of Lax pairs for this equation is as follows

$$W_n^{\{0\}} = \begin{pmatrix} -\dfrac{1}{2} & \dfrac{1+r_{n-1}}{1+s_{n-1}+r_{n-1}s_{n-1}} \\ \dfrac{s_n}{1+s_n+r_ns_n}\lambda & -\dfrac{1}{2} \end{pmatrix}, \tag{30}$$

when $m = 1$, system (28) reduces to

$$\begin{cases} r_{n_{t_1}} = -\dfrac{1+r_{n-2}}{(1+s_{n-2}+r_{n-2}s_{n-2})(1+s_{n-1}+r_{n-1}s_{n-1})^2} + \dfrac{(1+r_{n-1})^2s_n}{(1+s_n+r_ns_n)(1+s_{n-1}+r_{n-1}s_{n-1})^2}, \\ s_{n_{t_1}} = \dfrac{s_{n+2}}{(1+s_{n+2}+r_{n+2}s_{n+2})(1+s_{n+1}+r_{n+1}s_{n+1})^2} - \dfrac{(1+r_n)s_{n+1}^2}{(1+s_n+r_ns_n)(1+s_{n+1}+r_{n+1}s_{n+1})^2}. \end{cases}$$

Similarly, the Hamiltonian structures of discrete system (28), by using (26), can be established as follows

$$U_{n_{t_m}} = \begin{pmatrix} r_n \\ s_n \end{pmatrix}_{t_m} = J_2 \frac{\delta \hat{H}_n^m}{\delta u_n} = J_2 \Psi^m \begin{pmatrix} C_n^{(m)} \\ B_{n+1}^{(m)} \end{pmatrix} = J_2 \Psi^m \begin{pmatrix} \dfrac{s_n}{1+s_n+r_n s_n} \\ \dfrac{1+r_{n+1}}{1+s_{n+1}+r_{n+1}s_{n+1}} \end{pmatrix}. \tag{31}$$

From (26), we obtain the recurrence operator Ψ as follows

$$\Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix},$$

where

$$\Psi_{11} = \frac{E}{1+s_n+r_n s_n} + \frac{s_n}{1+s_n+r_n s_n}(1+E)(1-E)^{-1}\frac{1+r_n}{1+s_n+r_n s_n}E,$$
$$\Psi_{12} = -\frac{s_n}{1+s_n+r_n s_n}(1+E)(1-E)^{-1}\frac{s_n}{1+s_n+r_n s_n}E^{-1},$$
$$\Psi_{21} = -\frac{1+r_n}{1+s_n+r_n s_n}(1+E)(1-E)^{-1}\frac{r_n}{1+s_n+r_n s_n}E,$$
$$\Psi_{22} = \frac{E^{-1}}{1+s_n+r_n s_n} - \frac{1+r_n}{1+s_n+r_n s_n}(1+E)(1-E)^{-1}\frac{s_n}{1+s_n+r_n s_n}E^{-1}.$$

Let

$$K = J_2\Psi = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \tag{32}$$

with

$$K_{11} = -E^{-1}\frac{1+r_n}{1+s_n+r_n s_n}(1+E)(1-E)^{-1}\frac{1+r_n}{1+s_n+r_n s_n}E,$$
$$K_{12} = -E^{-1}\frac{1}{1+s_n+r_n s_n}E^{-1} + E^{-1}\frac{1+r_n}{1+s_n+r_n s_n}(1+E)(1-E)^{-1}\frac{s_n}{1+s_n+r_n s_n}E^{-1},$$
$$K_{21} = E\frac{1}{1+s_n+r_n s_n}E + E\frac{s_n}{1+s_n+r_n s_n}(1+E)(1-E)^{-1}\frac{1+r_n}{1+s_n+r_n s_n}E,$$
$$K_{22} = -E\frac{s_n}{1+s_n+r_n s_n}(1+E)(1-E)^{-1}\frac{s_n}{1+s_n+r_n s_n}E^{-1}.$$

It is easy to verify that K is a skew-symmetric operator.Hence, similarly, we obtain the following assertions:

Proposition 2. $\{\tilde{H}_n^{(m+1)}\}_{m\geq 0}$ *defined by* (31) *forms an infinite set of conserved functionals of the hierarchy* (28), *and* $\{\tilde{H}_n^{(m+1)}\}_{m\geq 0}$ *are in involution in pairs with respect to the Poisson bracket* (22).

Theorem 2. *The lattice soliton equations in* (28) *or the discrete Hamiltonian equations in* (31) *are all discrete Liouville integrable systems.*

3. The conservation laws

Consider the following Lax pair

$$\psi_{n+1} = \begin{pmatrix} \lambda & 1+r_n \\ \lambda s_n & 1+s_n+r_n s_n \end{pmatrix}\psi_n,$$
$$\psi_{n_t} = \begin{pmatrix} \frac{1}{2}\lambda - (1+r_n)s_{n-1} & 1+r_n \\ s_{n-1}\lambda & -\frac{1}{2}\lambda + (1+r_n)s_{n-1} \end{pmatrix}\psi_n. \tag{33}$$

Direct calculation yields

$$\psi_{n+1}^1 = \lambda\psi_n^1 + (1+r_n)\psi_n^2, \qquad \psi_{n+1}^2 = \lambda s_n \psi_n^1 + (1+s_n+r_n s_n)\psi_n^2,$$

and

$$\frac{\psi_{n+1}^1}{\psi_n^1} = \lambda + (1+r_n)\theta_n, \qquad \frac{\psi_{n+1}^2}{\psi_n^2} = \lambda s_n \theta_n^{-1} + (1+s_n+r_n s_n) \tag{34}$$

where $\theta_n = \frac{\psi_n^2}{\psi_n^1}$. From (34), we have

$$\lambda s_n = \lambda\theta_{n+1} + (1+r_n)\theta_n\theta_{n+1} - (1+s_n+r_ns_n)\theta_n. \tag{35}$$

From (33) and (34), a direct calculation gives

$$[\ln(\lambda+(1+r_n)\theta_n)]_t = (E-1)\left((1+r_n)\theta_n + \frac{1}{2}\lambda - (1+r_n)s_{n-1}\right). \tag{36}$$

Assume that

$$\theta_n = \sum_{j=1}^{\infty} \frac{\theta_n^{(j)}}{\lambda^{j-1}}. \tag{37}$$

From (35), we obtain the following recursion relation

$$\theta_{n+1}^{(j+1)} = (1+s_n+r_ns_n)\theta_n^{(j)} + s_n - (1+r_n)\sum_{i=1}^{j}\theta_n^{(i)}\theta_{n+1}^{(j-i+1)}. \tag{38}$$

Here the first terms in (38) are as follows

$$\begin{aligned}&\theta_n^{(1)} = s_{n-1}, \qquad \theta_n^{(2)} = s_{n-2},\\ &\theta_n^{(3)} = s_{n-3} - (1+r_{n-1})s_{n-2}^2, \ldots .\end{aligned} \tag{39}$$

Substituting (37) into (36), we obtain

$$\left[\ln\left((1+r_n)\sum_{j=1}^{\infty}\frac{\theta_n^{(j)}}{\lambda^{j-1}}\right)\right]_t = (E-1)\left((1+r_n)\sum_{j=1}^{\infty}\frac{\theta_n^{(j)}}{\lambda^{j-1}}\right). \tag{40}$$

Equating the power of $1/\lambda^2$ in (40), we can get an infinite number of conservation laws for the lattice soliton equation (15). The first two conservation laws are pointed out as follows

$$\begin{aligned}&(s_{n-2}+r_ns_{n-2})_t = (1+r_n)s_{n-1}(E-1)(s_{n-2}+r_ns_{n-2}),\\ &((1+r_n)s_{n-3} - (1+r_n)(1+r_{n-1})s_{n-2}^2)_t\\ &\quad = (1+r_n)(s_{n-1}(E-1)(1+r_n)(s_{n-3}-s_{n-2}^2-r_{n-1}s_{n-2}^2) + s_{n-2}(E-1)(1+r_n)s_{n-2}).\end{aligned} \tag{41}$$

Similarly, we can get the conservation laws of other lattice equation in the hierarchy (13).

4. Darboux transformation

In this section, a Darboux transformation (DT) for lattice (a) will be established. When taking $m = 1$ in (6) the spectral problem can be written as

$$Y_{n+1} = U_nY_n, \qquad Y_{n_t} = V_nY_n, \tag{42}$$

where

$$U_n = \begin{pmatrix}\lambda & 1+r_n\\ \lambda s_n & 1+s_n+r_ns_n\end{pmatrix}, \qquad V_n = \Gamma_n^{(1)} = \begin{pmatrix}\frac{1}{2}\lambda - (1+r_n)s_{n-1} & 1+r_n\\ \lambda s_{n-1} & -\frac{1}{2}\lambda + (1+r_n)s_{n-1}\end{pmatrix}. \tag{43}$$

Set $\phi_n = (\phi_n^1, \phi_n^2)^{\mathrm{T}}$, $\psi_n = (\psi_n^1, \psi_n^2)^{\mathrm{T}}$ are two basic solutions of (43) by use of (ϕ, ψ). We define the transformation matrix

$$T_n = \begin{pmatrix}\lambda + t_{11}(n) & t_{12}(n)\\ \lambda t_{21}(n) & \lambda + t_{22}(n)\end{pmatrix}, \tag{44}$$

with

$$\begin{aligned}&t_{11}(n) = \frac{\lambda_1\sigma_2(n) - \lambda_2\sigma_1(n)}{\sigma_1(n)-\sigma_2(n)}, \qquad t_{12}(n) = \frac{\lambda_2-\lambda_1}{\sigma_1(n)-\sigma_2(n)},\\ &t_{21}(n) = \frac{\sigma_1(n)\sigma_2(n)(\lambda_1-\lambda_2)}{\lambda_2\sigma_1(n)-\lambda_1\sigma_2(n)}, \qquad t_{22}(n) = \frac{(\sigma_2(n)-\sigma_1(n))\lambda_1\lambda_2}{\lambda_2\sigma_1(n)-\lambda_1\sigma_2(n)}\end{aligned} \tag{45}$$

and

$$\sigma_i(n) = \frac{\phi_n^2(\lambda_i) - \gamma_i \psi_n^2(\lambda_i)}{\phi_n^1(\lambda_i) - \gamma_i \psi_n^1(\lambda_i)}, \quad (i = 1, 2), \tag{46}$$

here λ_i, γ_i $(i = 1, 2)$ are the proper parameters and the terms in (45) and (46) are not zero. From (43) and (45), we have

$$\sigma_i(n+1) = \frac{\mu_i(n)}{\nu_i(n)}, \quad i = 1, 2, \tag{47}$$

here

$$\begin{aligned} \mu_i(n) &= \lambda_i s_n + (1 + r_n s_n)\sigma_i(n), \\ \nu_i(n) &= \lambda_i + r_n \sigma_i(n). \end{aligned} \tag{48}$$

From (47), a direct calculation yields

$$\det T_n = (\lambda - \lambda_1)(\lambda - \lambda_2). \tag{49}$$

Assume that there exists a gauge transformation

$$\tilde{Y}_n = T_n Y_n, \tag{50}$$

then the spectral problem (43) is transformed to

$$\tilde{U}_n = T_{n+1} U_n T_n^{-1}, \qquad \tilde{V}_n = (T_{nt} + T_n V_n) T_n^{-1} \tag{51}$$

from (47) and (48) we have

$$\begin{aligned} t_{11}(n+1) &= \frac{\mu_2(n)\nu_1(n)\lambda_1 - \mu_1(n)\nu_2(n)\lambda_2}{\mu_1(n)\nu_2(n) - \mu_2(n)\nu_1(n)}, \qquad t_{12}(n+1) = \frac{\nu_1(n)\nu_2(n)(\lambda_2 - \lambda_1)}{\mu_1(n)\nu_2(n) - \mu_2(n)\nu_1(n)}, \\ t_{21}(n+1) &= \frac{\mu_1(n)\mu_2(n)(\lambda_1 - \lambda_2)}{\mu_1(n)\nu_2(n)\lambda_2 - \mu_2(n)\nu_1(n)\lambda_1}, \qquad t_{22}(n+1) = \frac{(\mu_2(n)\nu_1(n) - \mu_1(n)\nu_2(n))\lambda_1\lambda_2}{\mu_1(n)\nu_2(n)\lambda_2 - \mu_2(n)\nu_1(n)\lambda_1}. \end{aligned} \tag{52}$$

Through direct but tedious calculations, from (46) and (52), we obtain the relations reading as

$$\begin{aligned} &(1 + r_n s_n) t_{12}(n+1) = t_{12}(n) t_{22}(n) + (1 + r_n)(t_{11}(n+1) - t_{22}(n)), \\ &t_{11}(n) t_{22}(n) = (t_{11}(n) + t_{22}(n) - t_{12}(n) t_{21}(n) - s_n t_{22}(n+1) \\ &\qquad - t_{21}(n+1) t_{22}(n) + (1 + r_n) t_{21}(n+1) t_{21}(n)) t_{22}(n+1). \end{aligned} \tag{53}$$

Hence, from all of the above statements, we obtain the following assertions

Proposition 3. *The matrix $\tilde{U}_n = T_{n+1} U_n T_n^{-1}$ has the same form as matrix U_n that is*

$$\tilde{U}_n = \begin{pmatrix} \lambda & 1 + \tilde{r}_n \\ \lambda \tilde{s}_n & 1 + \tilde{s}_n + \tilde{r}_n \tilde{s}_n \end{pmatrix} \tag{54}$$

in which the transformation formulae between old and new potentials are defined by

$$\tilde{r}_n = r_n - t_{12}(n), \qquad \tilde{s}_n = s_n + t_{21}(n+1). \tag{55}$$

The transformation $(\tilde{Y}_n; \tilde{r}_n, \tilde{s}_n \rightarrow Y_n; r_n, s_n)$ is called a Darboux transformation (DT) of the spectral problem (42), the transformation (55) is called a Bäcklund transformation (BT).

Proof. Let $T_n^{-1} = T_n^* / \det T_n$ and

$$T_{n+1} U_n T_n^* = \begin{pmatrix} f_{11}(\lambda, n) & f_{12}(\lambda, n) \\ f_{21}(\lambda, n) & f_{22}(\lambda, n) \end{pmatrix}. \tag{56}$$

It is easy to see that $f_{11}(\lambda, n)$, $\lambda f_{12}(\lambda, n)$, $f_{21}(\lambda, n)$ and $\lambda f_{22}(\lambda, n)$ are cubic-polynomials in λ. Also, we can readily verify that $f_{kl}(\lambda_i, n) = 0$ $(i, k, l = 1, 2)$. From (45), we have

$$\begin{aligned} &\lambda_i + t_{11}(n) + t_{12}(n)\sigma_i(n) = 0, \\ &\lambda_i t_{21}(n) + (\lambda_i + t_{22}(n))\sigma_i(n) = 0, \quad i = 1, 2. \end{aligned} \tag{57}$$

Based on the above results, we can suppose

$$T_{n+1} U_n T_n^* = (\det T_n) P_n,$$

with

$$P_n = \begin{pmatrix} p_{11}^0 + \lambda p_{11}^1 & p_{12}^0 \\ p_{21}^0 + \lambda p_{21}^1 & p_{22}^0 \end{pmatrix}. \tag{58}$$

That is

$$T_{n+1}U_n = P_nT_n, \tag{59}$$

where p_{ij}^l, $(i,j = 1,2; l = 0,1)$ are undetermined functions independent of λ. By comparing the coefficients of λ^i $(i = 0,1,2)$ in both sides of (59) we obtain

$$\begin{aligned} &p_{11}^1 = 1, \qquad p_{12}^0 = 1 + r_n - t_{12}(n) = 1 + \tilde{r}_n, \qquad p_{11}^0 = p_{21}^0 = 0, \qquad p_{21}^1 = s_n + t_{21}(n+1) = \tilde{s}_n, \\ &p_{22}^0 = (1+r_n)t_{21}(n+1) + 1 + s_n + r_ns_n - s_nt_{12}(n) + t_{12}(n)t_{21}(n) = 1 + \tilde{s}_n + \tilde{r}_n\tilde{s}_n. \end{aligned}$$

Thus we complete the proof. □

Proposition 4. *Under the gauge transformation (55) $\tilde{V}_n$ has the following form*

$$\tilde{V}_n = \begin{pmatrix} \frac{1}{2}\lambda - (1+\tilde{r}_n)\tilde{s}_{n-1} & 1 + \tilde{r}_n \\ \lambda\tilde{s}_{n-1} & -\frac{1}{2}\lambda + (1+\tilde{r}_n)\tilde{s}_{n-1} \end{pmatrix}. \tag{60}$$

Proof. Let $T_n^{-1} = T_n^*/\det T_n$ and

$$(T_{nt} + U_nV_n)T_n^* = \begin{pmatrix} g_{11}(\lambda,n) & g_{12}(\lambda,n) \\ g_{21}(\lambda,n) & g_{22}(\lambda,n) \end{pmatrix} \tag{61}$$

it is easy to see that $g_{11}(\lambda,n)$, $\lambda g_{12}(\lambda,n)$, $g_{21}(\lambda,n)$ and $g_{22}(\lambda,n)$ are cubic-polynomials in λ. Also, we can readily verify that $g_{kl}(\lambda_i,n) = 0$ $(i,k,l = 1,2)$, respectively. Taking (46) and (57) into account, it follows

$$\begin{aligned} &t_{11_t}(n) + \sigma_i(n)t_{12_t}(n) + \sigma_{i_t}(n)t_{12}(n) = 0, \\ &\lambda_i t_{21_t}(n) + \sigma_i(n)t_{22_t}(n) + (\lambda_i + t_{22}(n))\sigma_i(n) = 0, \\ &\sigma_{i_t}(n) = \lambda s_{n-1} + (-\lambda_i + 2(1+r_n)s_{n-1})\sigma_i(n) - (1+r_n)\sigma_{i_t}(n)^2. \end{aligned} \tag{62}$$

Based on the above results, we can suppose

$$(T_{nt} + U_nV_n)T_n^* = (\det T_n)G_n, \tag{63}$$

with

$$G_n = \begin{pmatrix} g_{11}^0 + \lambda g_{11}^1 & g_{12}^0 \\ g_{21}^0 + \lambda g_{21}^1 & g_{22}^0 + \lambda g_{22}^1 \end{pmatrix}. \tag{64}$$

That is

$$T_{nt} + U_nV_n = G_nT_n, \tag{65}$$

where r_{ij}^l, $(i,j = 1,2; l = 0,1)$ are undetermined functions independent of λ. By comparing the coefficients of λ^i $(i = 0,1,2)$ in both sides of (63) noticing (43) we have

$$\begin{aligned} &g_{11}^1 = \frac{1}{2}, \qquad g_{12}^0 = 1 + r_n - t_{12}(n) = 1 + \tilde{r}_n, \qquad g_{21}^1 = s_{n-1} + t_{21}(n) = \tilde{s}_{n-1}, \qquad g_{22}^1 = -\frac{1}{2}, \\ &g_{21}^0 = 0, \qquad g_{11}^0 = -(1 + r_n - t_{12}(n))(s_{n-1} + t_{21}(n)) = -(1+\tilde{r}_n)\tilde{s}_{n-1}, \\ &g_{22}^0 = (1 + r_n - t_{12}(n))(s_{n-1} + t_{21}(n)) = (1+\tilde{r}_n)\tilde{s}_{n-1}. \end{aligned}$$

The proof is thus completed. □

From the above propositions we come to the following proposition

Proposition 5. *Every solution r_n, s_n of (a) is mapped into a new solution $\tilde{r}_n, \tilde{s}_n$ under the BT (55). Homoplastically, noticing that (a) and (b) are based on the same eigenvalue problem (6), we can construct a homeotypic Darboux matrix with (57) for (b). So we arrive at an assertion for (b).*

Proposition 6. *The transformation* $(\tilde{Y}_n; \tilde{r}_n, \tilde{s}_n \to Y_n; r_n, s_n)$ *is the Darboux transformation of the spectral problem* (43) *under the transformation*

$$\tilde{r}_n = r_n - t_{12}(n), \qquad \tilde{s}_n = s_n + t_{21}(n+1), \tag{66}$$

the solution r_n, s_n *are mapped into the new solution* $\tilde{r}_n, \tilde{s}_n$.

From (15), substituting the trivial solution $r_n = 0, s_n = 1$ into (43) leads to

$$Y_{n+1} = \begin{pmatrix} \lambda & 1 \\ \lambda & 2 \end{pmatrix} Y_n, \qquad Y_{n_t} = \begin{pmatrix} \frac{1}{2}\lambda - 1 & 1 \\ \lambda & -\frac{1}{2}\lambda + 1 \end{pmatrix} Y_n, \tag{67}$$

its basic solutions can be chosen as

$$\begin{aligned} \phi_n &= \left(\frac{\lambda + 2 + \sqrt{\lambda^2+4}}{2}\right)^n \exp\left(\frac{\sqrt{\lambda^2+4}}{2}t\right) \begin{pmatrix} 1 \\ \frac{-\lambda + 2 + \sqrt{\lambda^2+4}}{2} \end{pmatrix}, \\ \psi_n &= \left(\frac{\lambda + 2 - \sqrt{\lambda^2+4}}{2}\right)^n \exp\left(-\frac{\sqrt{\lambda^2+4}}{2}t\right) \begin{pmatrix} 1 \\ \frac{-\lambda + 2 - \sqrt{\lambda^2+4}}{2} \end{pmatrix}. \end{aligned} \tag{68}$$

Substituting (68) into (46) we obtain

$$\sigma_i(n) = \frac{\omega_i^n e^{\sqrt{\lambda_i^2+4t}}\left(-\lambda_i + 2 + \sqrt{\lambda_i^2+4}\right) - \gamma_i\left(-\lambda_i + 2 - \sqrt{\lambda_i^2+4}\right)}{2\omega_i^n e^{\sqrt{\lambda_i^2+4t}} - 2\gamma_i}, \quad (i = 1, 2),$$

where $\omega_i = \frac{\lambda_i^2 + 2\lambda_i + 4 + (\lambda_i+2)\sqrt{\lambda_i^2+4}}{2\lambda_i}$, $i = 1, 2$, therefore the new solutions are given as follows

$$\begin{aligned} \tilde{r}_n &= -\frac{\lambda_2 - \lambda_1}{\alpha_1(n) - \alpha_2(n)}, \\ \tilde{s}_n &= 1 + \frac{(\lambda_1 + 2\alpha_1(n))(\lambda_2 + 2\alpha_2(n))(\lambda_1 - \lambda_2)}{\lambda_2(\lambda_1 + 2\alpha_1(n))(\lambda_2 + \alpha_2(n)) - \lambda_1(\lambda_1 + \alpha_1(n))(\lambda_2 + 2\alpha_2(n))}. \end{aligned} \tag{69}$$

Starting from the explicit solitons (69), we apply the Darboux transformation (50) and the Bäcklund transformation (55) once again, then new solitons of (13) and (28) are obtained. This process can be done continually. Therefore, we can obtain many explicit solitons for the lattice equation (13) and (28).

5. Conclusions and remarks

In this paper, starting from a new discrete isospectral problem, two hierarchies of nonlinear integrable differential–difference soliton equations are derived. It is shown that every equation in the resulting models is integrable in Liouville sense. It is also shown that these two hierarchies correspond to positive and negative power expansions with respect to spectral parameter. Moreover, the conservation laws of the obtained equation hierarchies are discussed. Further, with the help of the gauge transformations of Lax pairs, a Darboux transformation is established for the first nonlinear equations of resulting hierarchies, by means of which the exact solutions are given.

We should mention that this study, to some extent, presents us a design and implementation for new soliton hierarchies of lattice versions. We believe that the present studies can be useful for other applications and can be extended to more complicated spectral problems in higher order. Furthermore, we would emphasize the mathematical and physical background as well as the deeper properties, such as the symmetries, nonlinearization. The nice rational equations from positive hierarchy (19) would be exploited gradually on another occasion.

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